Limit shapes in the Schur process

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Outline

- Pyramid partitions
- Interlude into partitions and the Schur process
- Asymptotics of pyramid partitions
- Asymptotics of non–uniform Aztec diamonds
- Some related phenomena

Pyramid partitions

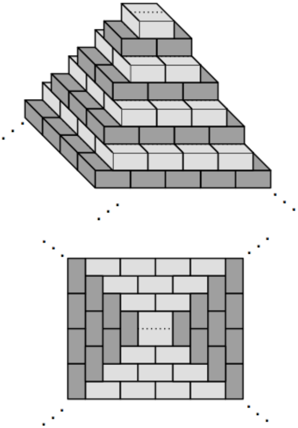


Figure : Piles of $2 \times 2 \times 1$ boxes, each viewed as a pair of dominoes in the 2D projection looking downwards. On the left, the *empty* pyramid partition.

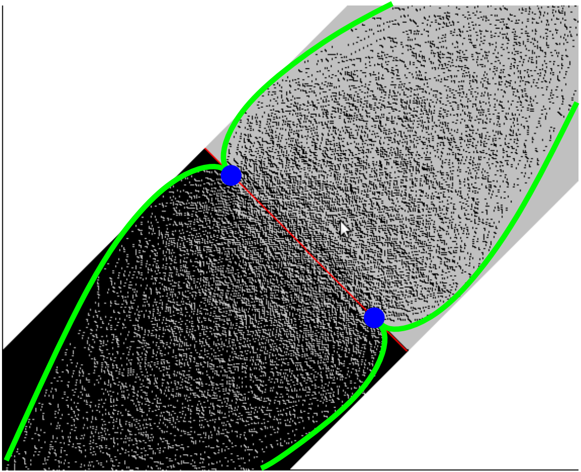
Flips and the volume

- pyramid partition = what's left after a finite number of box removals from the empty configuration (introduced by Kenyon and Szendröi)
- removal = flip (adjacent vertical dominoes $\leftrightarrow$ adjacent horizontal dominoes)
- Volume = Number of flips

Theorem (Young 2010)

$$\sum_{\Lambda} q^{Volume(\Lambda)} = \prod_{n \geq 1} \frac{(1+q^{2n-1})^{2n-1}}{(1-q^{2n})^{2n}}.$$

How do large pyramid partitions look like?



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Partitions

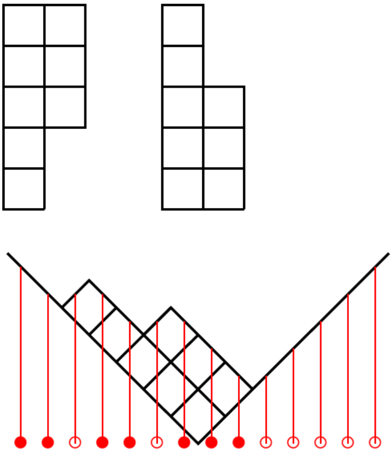

Figure : Partition $(2, 2, 2, 1, 1)$ in English, French and Russian notation, with associated Maya diagram (particle-hole representation).

Horizontal and vertical strips

Given partitions $\mu \subseteq \lambda$, we can form skew diagram λ/μ, which we call a

- horizontal strip, and write $\mu \prec \lambda$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \dots$$

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- vertical strip, and write $\mu \prec' \lambda$, if $\lambda' \prec \mu'$ ($' =$ conjugate) or

$$\lambda_i - \mu_i \in \{0, 1\}$$

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The Schur process

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \{\prec, \succ, \prec', \succ'\}^n$ be a word. We say a sequence of partitions $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$ is *ω-interlaced* if $\lambda(i-1)\,\omega_i\,\lambda(i)$, for $i = 1, \ldots, n$. The *Schur process* of word ω with parameters $Z = (z_1, \ldots, z_n)$ is the measure on the set of ω-interlaced sequences of partitions

$$\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$$

given by

$$Prob(\Lambda) \propto \prod_{i=1}^{n} z_i^{||\lambda(i)| - |\lambda(i-1)||}.$$

Remark

For a more general definition, see the original work of Okounkov–Reshetikhin 2003, or Borodin–Rains 2006.

The Schur process is a determinantal point process

Theorem (OR 2003; BR 2006)

$$\textit{Prob}(\lambda(i_s) \textit{ contains a particle at position } k_s, 1 \leq s \leq n) = \det_{1 \leq u,v \leq n} K(i_u, k_u; i_v, k_v)$$

where

$$K(i,k;i',k') = \begin{cases} \left[\frac{z^k}{w^{k'}}\right] \frac{\Phi(z;Z,\omega;i)}{\Phi(w;Z,\omega;i')} \frac{\sqrt{zw}}{z-w}, & i \leq i', \\ -\left[\frac{z^k}{w^{k'}}\right] \frac{\Phi(z;Z,\omega;i')}{\Phi(w;Z,\omega;i)} \frac{\sqrt{zw}}{w-z}, & i > i' \end{cases}$$

with

$$\Phi(z;Z,\omega;i) = \prod_{\substack{j:\ j \leq i,\ \omega_j \in \{\prec, \prec'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \prec', \\ -1, & \omega_j = \prec. \end{cases}}} (1 + \epsilon_j z_j z)^{\epsilon_j} \prod_{\substack{j:\ j > i,\ \omega_j \in \{\succ, \succ'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \succ', \\ -1, & \omega_j = \succ. \end{cases}}} \left(1 + \epsilon_j \frac{z_j}{z}\right)^{-\epsilon_j}$$

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Pyramid partitions as Schur processes, pictorially

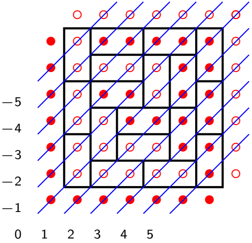


Figure : A pyramid partition of width 5 corresponding to the sequence
$\emptyset \prec (1) \prec' (2) \prec (2,2) \prec' (3,3) \prec (3,3,2) \succ' (2,2,1) \succ (2,1) \succ' (1,1) \succ (1) \succ' \emptyset$.

Pyramid partitions as Schur processes II

Let $n = 2n_0$ be an even integer. A pyramid partition is (bijectively) a sequence of $2n+1$ partitions

$$\Lambda = (\emptyset = \lambda(-n) \prec \lambda(-n+1) \prec' \lambda(-n+2) \prec \cdots \prec' \lambda(0) \succ \lambda(1) \succ' \lambda(2) \succ \cdots \succ' \lambda(n) = \emptyset).$$

It is this a Schur process for the word $\omega_{pyr} = (\prec, \prec')^{n_0} (\succ, \succ')^{n_0}$ and parameters $Z = (z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n)$.

Remark

For volume weighting weighting, $z_{-i} = z_i = q^{i-\frac{1}{2}},\ 1 \leq i \leq n$.

A simple word on asymptotics

Everything we'd like to know about asymptotics of large pyramid partitions can be translated into asymptotics of large particle–hole systems associated to the corresponding Schur process.

How to compute the limit shape

Let $t = 2t_0 < n$, $k \in \mathbb{Z} + \frac{1}{2}$. A weak Wick lemma shows that:

Lemma (db–Boutillier–Vuletić 2015)

$$\begin{aligned}
&Prob(\lambda(-t) \text{ contains a particle at position } k) = \\
&= \left[\frac{z^k}{w^k}\right] \frac{J(z;t_0)}{J(w;t_0)} \frac{\sqrt{zw}}{z-w} \\
&= \int\int \frac{J(z;t_0)}{J(w;t_0)} \frac{1}{z^{k-\frac{1}{2}} w^{-k-\frac{1}{2}}} \frac{1}{z-w} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w}
\end{aligned}$$

where (with $(u;q)_m = \prod_{i=0}^{m-1}(1-q^i u)$*)*

$$J(z;t_0) = \frac{(-q^{2t_0+\frac{1}{2}}z;q^2)_{n_0-t_0}(\frac{q^{\frac{1}{2}}}{z};q^2)_{n_0}}{(q^{2t_0+\frac{3}{2}}z;q^2)_{n_0-t_0}(-\frac{q^{\frac{3}{2}}}{z};q^2)_{n_0}}.$$

Asymptotics regime

We let the size of the partition grow with $q \to 1$ as $\epsilon \to 0$ like so:

$$\begin{aligned}
q(\epsilon) &= \exp(-\gamma\epsilon), \\
n_0(\epsilon) &= a_0/\epsilon, \\
t_0(\epsilon) &= x_0/\epsilon, \\
k(\epsilon) &= y/\epsilon.
\end{aligned}$$

A few limit formulas

If $q = \exp(-r)$ and $r \to 0+$, we have

$$\log(z;q)_\infty \sim -\frac{Li_2(z)}{r}$$

and furthermore,

$$\log(z;q)_{\frac{A}{r}} \sim \frac{1}{r}(Li_2(e^{-A}z) - Li_2(z))$$

where

$$Li_2(z) = \sum_{n\geq 1} \frac{z^2}{n^2}, \ |z| < 1$$

with analytic continuation given by

$$Li_2(z) = -\int_0^z \frac{\log(1-u)}{u} du, \quad z \in \mathbb{C}\backslash[1,\infty).$$

Asymptotics of the kernel

Lemma (db–Boutillier–Vuletić 2015)

In the limit ($x = 2x_0$ is rescaled t, y is rescaled k),

$$Prob(\lambda(-t) \text{ contains a particle at position } k) \sim \int\int e^{S(z;x,y)-S(w;x,y))} \frac{d\mathbb{T}}{z-w}$$

where

$$S(z;x,y) = \frac{1}{2\gamma}\Big(Li_2(-Az) - Li_2(-Xz) + Li_2(\frac{A}{z}) - Li_2(\frac{1}{z}) + \\ + Li_2(Xz) - Li_2(Az) + Li_2(-\frac{1}{z}) - Li_2(-\frac{A}{z})\Big) - y\log z$$

and $X = \exp(-\gamma x), A = \exp(-2\gamma a_0)$.

The arctic curve

To compute the arctic curve, one solves for (x, y) (or $X = \exp(-\gamma x)$, $Y = \exp(2\gamma y)$) corresponding to double critial points of $S(z; x, y)$. That is,

Theorem (db–Boutillier–Vuletić 2015)

The arctic curve is the locus (x, y) *satisfying:*

$$\begin{aligned} f(z; X) &= Y, \\ f'(z; X) &= 0 \end{aligned}$$

where $f(z; X) = \frac{(z+1)(z-A)(z-1/A)(z+1/X)}{(z-1)(z+A)(z+1/A)(z-1/X)}$.

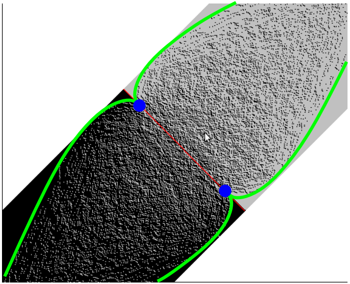
Remark

Alternatively, it can be seen as given by the *algebraic* equation

$$\Delta\left[(z+1)(z-A)(z-1/A)(z+1/X) - Y(z-1)(z+A)(z+1/A)(z-1/X)\right] = 0$$

where Δ represents taking the discriminant.

The arctic curve, pictorially



Notice the cusps (which correspond to the *triple* critical point of S at $z = 0$). This cusp phenomenon has appeared in the case of (skew) plane partitions with two different q's, Mkrtchyan 2013.

Intuitively, we have replaced "two different q's, word $\omega = \prec^{2n_0} \succ^{2n_0}$" with "one single q, word $\omega = (\prec, \prec')^{n_0}(\succ, \succ')^{n_0}$". If this makes no sense, it's probably because it doesn't make much sense.

Arctic curve in the infinite regime

What happens when $a_0 \to \infty$, or equivalently, $A \to 0$?

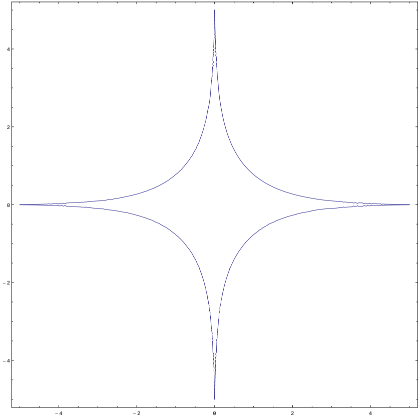
The cusps move to ∞ and the arctic curve becomes

$$(1 + Z + W - ZW)(1 + Z - W + ZW)(1 - Z + W + ZW)(1 - Z - W - ZW) = 0$$

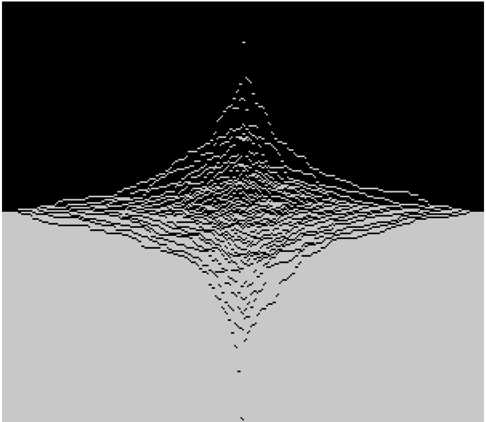
where $(Z, W) = (\sqrt{X}, \sqrt{Y})$ which is the boundary of the amoeba of the (square lattice determined) polynomial

$$P(Z, W) = 1 + Z + W - ZW.$$

Arctic curve in the infinite regime, pictorially



A large sample in the infinite regime, up to affine transformations



A word on fluctuations around the arctic curve

Everywhere but at the cusps, fluctuations are of Airy type (cf., for example, Okounkov–Reshetikhin 2006). At the cusps, because of the appearence of the triple critical point, one would conjecture Pearcey process fluctuations, but this has not yet been rigurously established.

A slide on details: vertex operators

$$\Gamma_+(x)\Gamma_-(y) = \frac{1}{1-xy}\Gamma_-(y)\Gamma_+(x),$$

$$\tilde{\Gamma}_+(x)\tilde{\Gamma}_-(y) = \frac{1}{1-xy}\tilde{\Gamma}_-(y)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_+(x)\Gamma_-(y) = (1+xy)\Gamma_-(y)\tilde{\Gamma}_+(x),$$

$$\Gamma_+(x)\tilde{\Gamma}_-(y) = (1+xy)\tilde{\Gamma}_-(y)\Gamma_+(x),$$

$$\Gamma_+(x)\psi(z) = \frac{1}{1-xz}\psi(z)\Gamma_+(x),$$

$$\Gamma_+(x)\psi^*(w) = (1-xw)\psi^*(w)\Gamma_+(x),$$

$$\Gamma_-(y)\psi(z) = \frac{1}{1-\frac{y}{z}}\psi(z)\Gamma_-(y),$$

$$\Gamma_-(y)\psi^*(w) = (1-\frac{y}{w})\psi^*(w)\Gamma_-(y),$$

$$\tilde{\Gamma}_+(x)\psi(z) = (1+xz)\psi(z)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_+(x)\psi^*(w) = \frac{1}{1+xw}\psi^*(w)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_-(y)\psi(z) = (1+\frac{y}{z})\psi(z)\tilde{\Gamma}_-(y),$$

$$\tilde{\Gamma}_-(y)\psi^*(w) = \frac{1}{1+\frac{y}{w}}\psi^*(w)\tilde{\Gamma}_-(y).$$

Other stuff: "skew pyramid partitions"

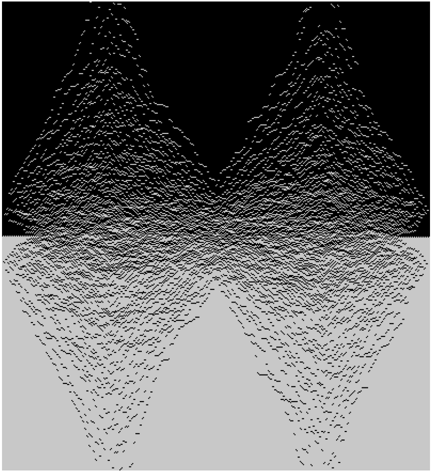
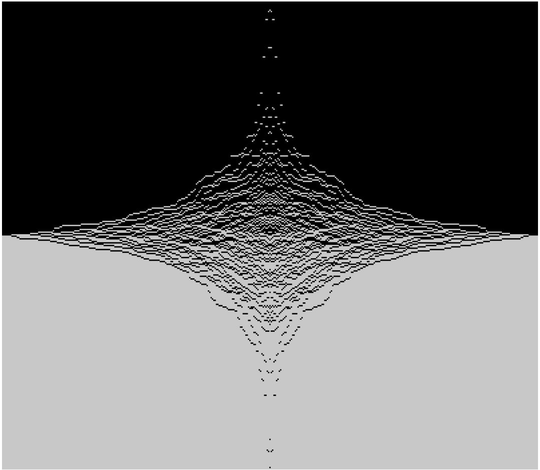
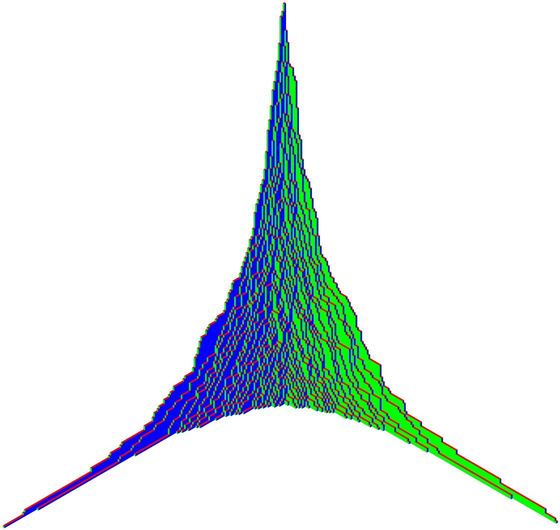


Figure : Skew pyramid partitions: word $(\prec, \prec')^{50}, (\succ, \succ')^{50}, (\prec, \prec')^{50}, (\succ, \succ')^{50}$, $q = 0.99$. The analogue in pyramid partition land of OR 2006's skew plane partitions. Vertical cusps should have Pearcey fluctuations.

Other stuff: symmetric “pyramid partitions”



Symmetric "pyramid partitions" as plane overpartitions



This limit shape seems to be the same that Vuletić 2009 analyzed in the context of strict plane partitions.

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The Aztec diamond

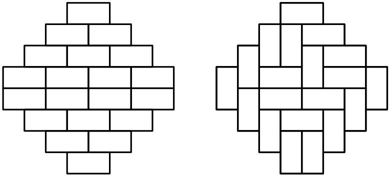
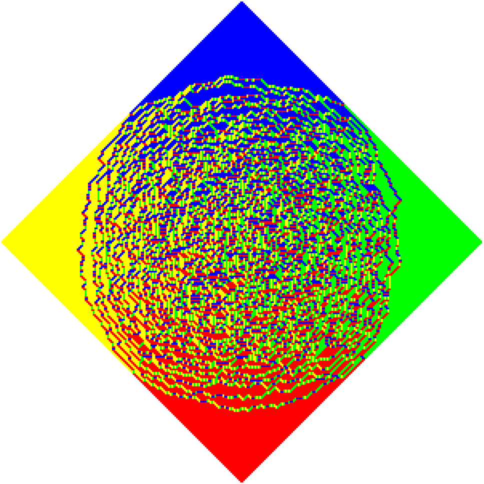


Figure : Two tilings of the size $n = 4$ Aztec diamond. One can define the volume of a tiling as the number of flips needed to reach it from the all horizontal (zero volume) tiling.

There are $2^{\binom{n+1}{2}}$ tilings of the $n \times n$ Aztec diamond (Elkies–Kuperberg–Larsen–Propp 1992).

The original arctic circle theorem (Jockush–Propp–Shor 1998)



The Aztec diamond as a Schur process

The order n Aztec diamond is (bijectively equivalent to) the sequence of $2n+1$ partitions

$$\Lambda = (\emptyset = \lambda(0) \prec \lambda(1) \succ' \lambda(2) \prec \cdots \succ' \lambda(2n-1) \prec \lambda(n) \succ' \lambda(2n) = \emptyset).$$

It is a Schur process of word $(\prec, \succ')^n$ and parameters $(z_1, \ldots, z_n)$.

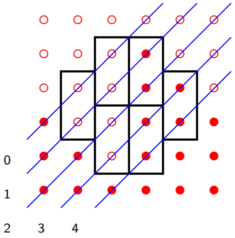


Figure : A 2×2 Aztec diamond corresponding to the sequence $\emptyset \prec (2) \succ' (1) \prec (1,1) \succ' \emptyset$.

Remark

If $z_{2i-1} = q^{-2i+1}, z_{2i} = q^{2i}$, one obtains a q^{Volume} weighting on the Aztec diamond where volume $=$ number of flips from the all horizontal tiling.

Periodic weightings, arbitrary parameters

For (say) $k < l$, pick z parameters as follows: $z_1 = a_1, z_2 = b_1, z_3 = a_2, z_4 = b_2, \ldots, z_{2k-1} = a_k, z_{2k} = b_k, z_{2k+1} = a_1, z_{2k+2} = b_{k+1}, \ldots, z_{2l} = b_l$, *repeat*

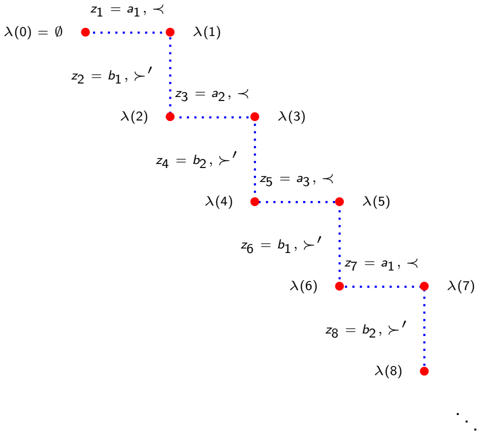


Figure : $k = 3, l = 2$ a, b parameters.

Such weights have been considered before (Mkrtchyan 2013, case of plane partitions), but note here there is no need for any of the parameters to be < 1.

How do such large Aztec diamonds look?

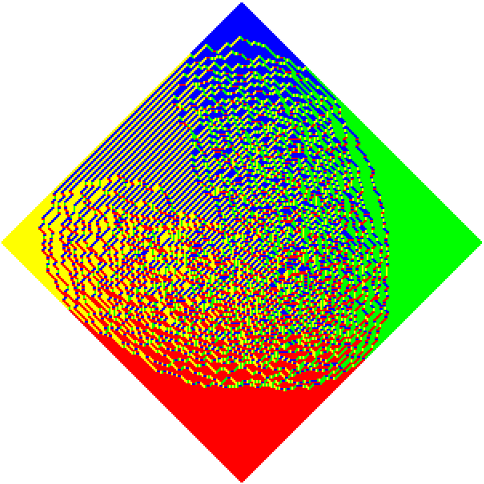


Figure : A random 150×150 Aztec diamond with a, b parameters $a_1 = 4$, $a_2 = 1/4$, $b_1 = 1$.

More fingers

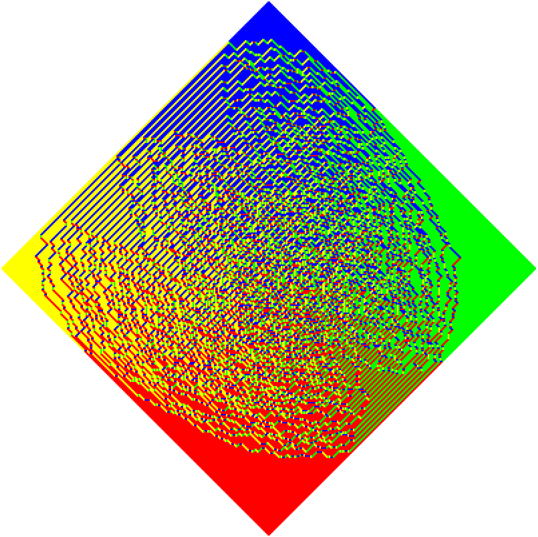


Figure : A random 200×200 Aztec diamond with a, b parameters $a_1 = 8, a_2 = 1, a_3 = 1/8, b_1 = 3, b_2 = 1/3$.

Snake

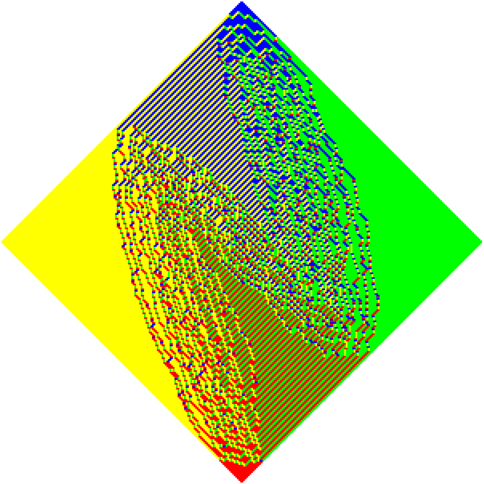


Figure : A random 150×150 Aztec diamond with a, b parameters $a_1 = 48, a_2 = 1, a_3 =, b_1 = 16, b_2 = 1/8$.

Compare with Kenyon–Okounkov 2003 (another snake)

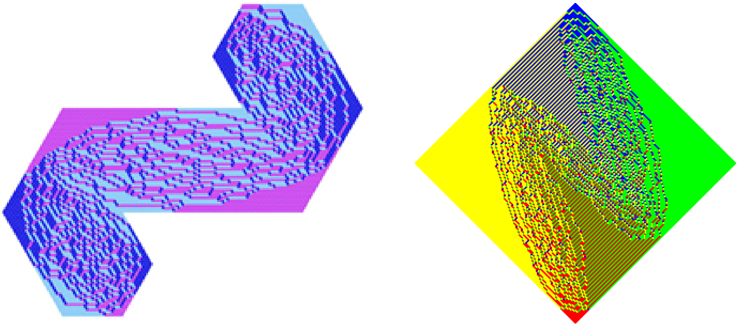


Figure : Simple (uniform) measure and (moderately) complicated boundary conditions vs. (moderately) complicated measure and simple boundary conditions (ignoring the elephant in the room: that the two lattices are different).

Something like the tacnode process

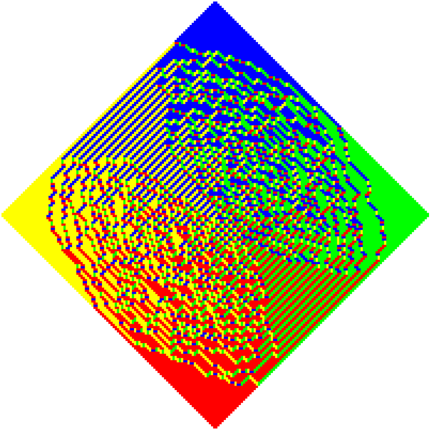
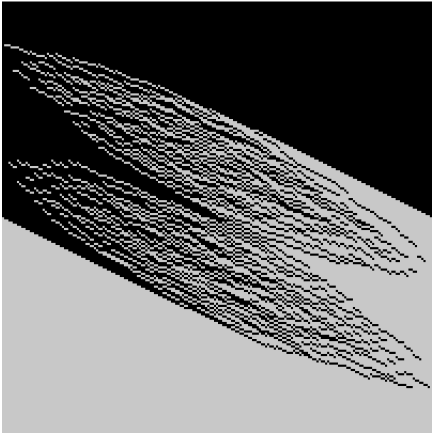


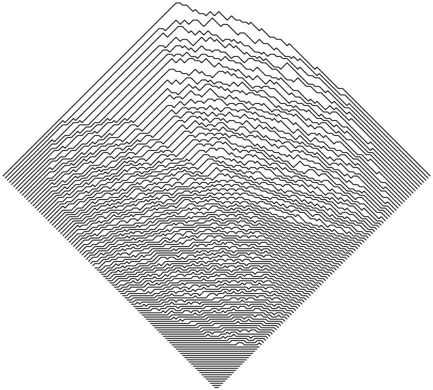
Figure : 100×100 Aztec diamond with a, b parameters $(a_1, a_2) = (b_1, b_2) = (\alpha, 1/\alpha), \alpha = 30$.

Particle–hole view, up to affine transformations



Compare this to the work of Borodin–Duits 2011.

Non–intersecting paths picture



The S function, and what can we say 'bout the model

$$S(z;x,y) = \frac{x}{k}\log\left(\prod_{i=1}^{k}(1+a_i z)\right) + (1-\frac{x}{l})\log\left(\prod_{i=1}^{l}(1-\frac{b_i}{z})\right) - y\log z$$

Analyzing S, we can obtain:

- formula for the arctic curve, as before
- location of the points of tangency to the boundary
- angle made by the cusps
- fluctuations (which ought to be as before)

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Some pictures of things we can't yet do using our methods. Some are harder than others.

Partial Aztec diamonds, uniformly weighted

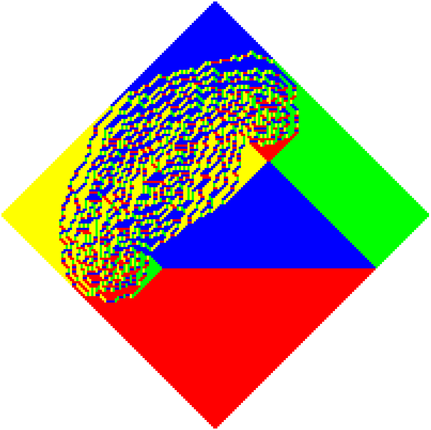
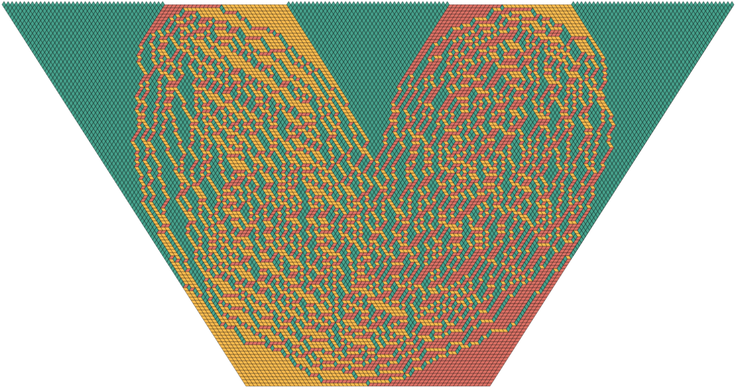


Figure : Half an $n = 100$ Aztec diamond with final partition fixed $\emptyset \prec \lambda(1) \succ' \cdots \prec \lambda(n-1) \succ' \lambda(50) = 25^{50}$

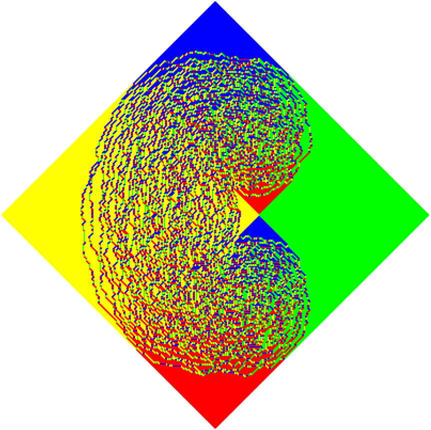
This corresponds to the Gelfand–Tsetlin polygons of Petrov 2012 (special case of Kenyon–Okounkov 2007). In our case:

$$\emptyset \prec \lambda(1) \succ' \cdots \prec \lambda(n-1) \succ' \lambda(n) = \textit{fixed } \lambda.$$

Compare with GT polygons – KO 2007 and P 2012

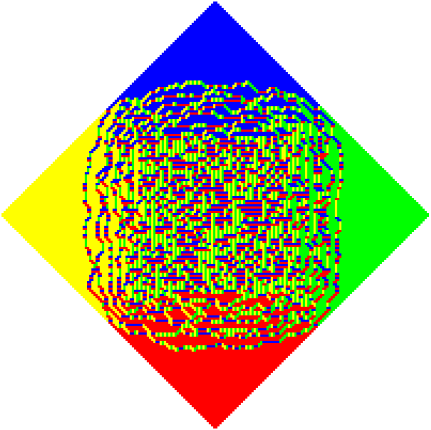


Aztec diamond with frozen corner

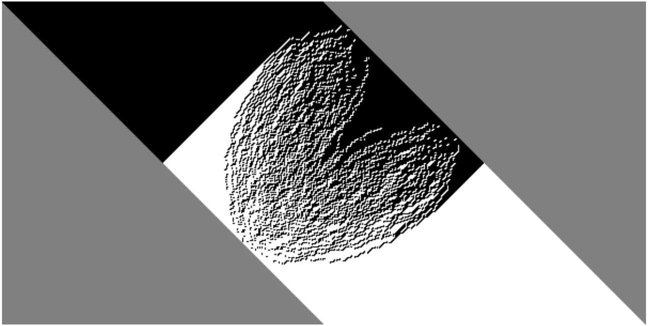


Work of Colomo–Sportiello, initially on the 6 vertex model. Ask Filippo and/or Andrea. There should be some Painlevé interpretation of the partition function here, a la Borodin–Arinkin 2009.

2–Periodic Aztec diamond



Studied by Chhita–Johansson 2014 and Chhita–Young 2013 using the inverse Kasteleyn matrix approach.



Thank you!